gas (constant γ) and to blunt bodies with sonic corners. The basic inputs to the method of Ref. 5 are M_∞, γ and model geometry. The broken lines of Fig. 2 represent shock shapes calculated for M_∞ and $\gamma_{E,\infty}$ as basic inputs; the solid lines denote M_∞ and $\gamma_{E,2}$ as basic inputs; the dashed lines denote M_∞ and γ_{eff} as basic inputs. The γ_{eff} was determined from the ideal-gas, normal shock relation

$$\gamma_{\text{eff}} = [1+\varepsilon(1-2/M_\infty{}^2)]/(\varepsilon-1)$$

where M_∞ and ε are real-gas, equilibrium values obtained from Ref. 2.

The expected decrease in measured shock detachment distance with increasing ε is illustrated in Fig. 2, with Δ decreasing by a factor of 2.5 as ε increases from 4 to 19. Relatively good agreement exists between measured and calculated shock shapes for helium as expected, since the present helium flow conditions correspond to ideal-gas behavior ($\gamma_{E,\infty} = \gamma_{E,2} = \gamma_{\text{eff}}$). For air and CO_2, the agreement between measured and calculated shock shape improves when $\gamma_{E,2}$, instead of $\gamma_{E,\infty}$, is used as an input to the program of Ref. 5. When γ_{eff} is used as input, the measured and calculated shock shapes are observed to be in good agreement. This result supports the proposal (for example, Ref. 6) that flowfield calculations for blunt bodies where real-gas effects are significant can be greatly simplified by use of ideal-gas relations with an appropriate γ.

In Fig. 3, measured and calculated shock shapes for the Viking aeroshell model are presented for the three test gases. This figure, in conjunction with Fig. 2, illustrates the effect of body shape on shock shape and detachment distance. The effect of body shape becomes more pronounced as ε increases. For example, the ratio of measured Δ for the flat-faced cylinder to measured Δ for the aeroshell is approximately 1.9 for helium ($\varepsilon = 3.8$), 4 for air ($\varepsilon = 11.4$), and 7 for CO_2 ($\varepsilon = 18.9$). Again, the measured and calculated results for helium are observed to be in good agreement, and usage of γ_{eff} in the program of Ref. 5 improves agreement between measured and calculated shock shapes; however, this agreement for the Viking aeroshell is somewhat poorer than that for the flat-faced cylinder.

A problem characteristic of high-enthalpy facilities, such as the expansion tube, is determination of the thermochemical state of the flow. Results obtained in the Langley pilot model expansion tube with oxygen-nitrogen mixtures showed departure of the freestream flow from equilibrium (Ref. 7). Although nonequilibrium freestream effects on Δ should be small for air, freestream departure from equilibrium for CO_2 may be shown to decrease ε, thus increase standoff distance. For both test gases, nonequilibrium flow in the shock layer of the model will have an appreciable effect on Δ, with Δ for nonequilibrium flow being greater than that for equilibrium flow.[8] Additional studies are required to determine the possible extent of nonequilibrium in the postshock flow. However, the present tests suggest that for standoff distances of the order of those observed on the flat-faced cylinder model, nonequilibrium effects are relatively small; for the much smaller standoff distance of the Viking aeroshell model, some postshock nonequilibrium effects may be evident for CO_2. It should be noted, however, that a change in γ_{eff} for CO_2 of only 1.6% is required to bring the calculated shock displacement into agreement with measured shock displacement. Overall, the present results have verified that shock shapes observed in the hypersonic, real-gas flow of the expansion tube over a wide range of ε may be accurately described by simple, ideal-gas calculations (Ref. 5) provided γ_{eff} is used as input.

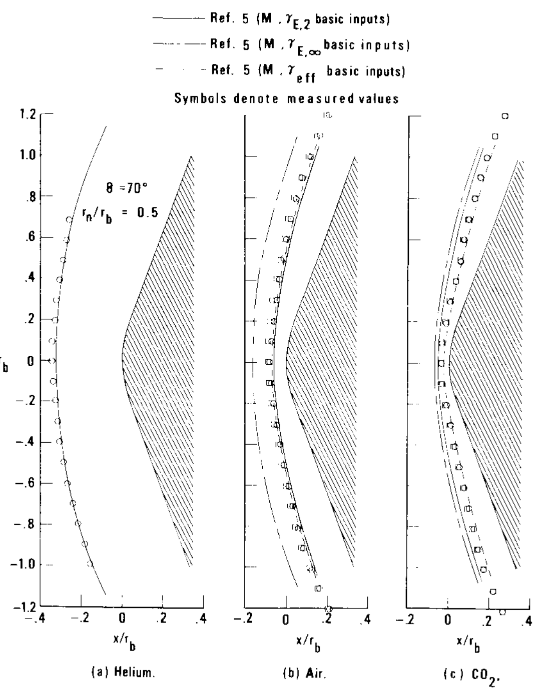

Fig. 3 Shock shapes for Viking aeroshell.

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Lift and Moment for Arbitrary Power-Law Upwash in Oscillating Subsonic Unsteady Thin-Airfoil Theory

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Introduction

IN oscillating unsteady airfoil theory, various types of chordwise upwash distributions are of interest, depending on the application. Classical flutter studies made use of constant and linear upwash distributions, which are appropriate to plunging and pitching airfoils. Sears[1] introduced the sinusoidal gust, and Kemp[2] introduced the generalized sinusoidal gust in which the time and space frequencies are different.

For incompressible flow, the lift and moment expressions for these upwashes are well known. The constant and linear results are in books on Aeroelesticity while the sinusoidal gust results

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are in Ref. 1. The generalized sinusoidal gust lift was given in Ref. 2, and the moment in Ref. 3.

Recently Osborne[4] put forward an approximate theory for oscillating thin airfoils in subsonic flow, and Kemp[5] was able to put Osborne's lift and moment expressions in closed form for all four types of upwash referred to above.

The purpose of this Note is to derive the lift and moment for an arbitrary integer power-law upwash

$$v = v_0 e^{ivt}(x/c)^n \tag{1}$$

by purely algebraic methods from the results for the generalized sinusoidal gust. Thus, the lift and moment for the upwash Eq. (1) will be given both for Osborne's approximate subsonic theory and for its incompressible limit, which is exact.

For values of $n > 1$, the response to a power-law upwash can be of interest in aeroelastic calculations, as in the study of flutter of deforming airfoils, for example.

Subsonic Flow

The upwash distribution for a generalized gust is

$$\begin{aligned} v(x,t) &= v_0 \exp(ivt - i\mu x/V) \\ &= v_0 \exp(ivt - i\lambda x/c) \end{aligned} \tag{2}$$

where c is the half-chord, and x is measured from the center of the airfoil toward the trailing edge.

In the approximate theory put forward by Osborne,[4] the following dimensionless parameters are used:

$$\lambda = \mu c/V, \quad \omega = vc/V, \quad \beta^2 = 1 - M^2$$
$$\omega^i = \omega/\beta^2, \quad \lambda^i = M^2\omega^i, \quad \Lambda = \lambda + \lambda^i \tag{3}$$

where M is the flight Mach number and V the flight speed. It is convenient to introduce the Bessel function abbreviations

$$J = J_0 - iJ_1, \quad C(z) = K_1(iz)/[K_0(iz) + K_1(iz)] \tag{4}$$

the second of which is the familiar Theodorsen function.

In these terms Kemp[5] has shown that the lift and nose-up moment associated with upwash [Eq. (2)] can be written in closed form as

$$L(t) = 2\pi\beta^{-1}\rho_\infty cVv_0 e^{ivt}\{J(\Lambda)[C(\omega^i)J(\lambda^i) + iJ_1(\lambda^i)] + i(\omega/\lambda)[J_0(\lambda^i)J_1(\Lambda) - J_1(\lambda^i)J_0(\Lambda)]\} \tag{5}$$

$$\begin{aligned} M(t) = \pi\beta^{-1}\rho_\infty c^2 Vv_0 e^{ivt}\{&J(\Lambda)[[C(\omega^i) - 1]][[J_0(\lambda^i) - J_2(\lambda^i)]] - \\ &2iJ_1(\lambda^i)C(\omega^i)] + (\omega/\lambda)[J_0(\Lambda)[[J_0(\lambda^i) - J_2(\lambda^i)]] + \\ &2J_1(\Lambda)J_1(\lambda^i)] - 2[(\lambda - \omega)/\lambda^2] \times \\ &[J_0(\Lambda)J_1(\lambda^i) - J_1(\Lambda)J_0(\lambda^i)]\} \end{aligned} \tag{6}$$

Now the upwash [Eq. (2)] can be expanded in a power series in λ as

$$v = v_0 e^{ivt} \sum_{n=0}^{\infty} (-i\lambda)^n (x/c)^n/n! \tag{7}$$

Thus the coefficient of $(-i\lambda)^n/n!$ is an integer power upwash distribution $(x/c)^n$. The lift and moment can also be expanded in powers of λ, and since they depend linearly on the upwash, we can expect that the coefficient of $(-i\lambda)^n/n!$ in the lift and moment expansions will be the lift and moment for the upwash distribution $(x/c)^n$. This expansion technique then permits the derivation of the forces for any positive integer power upwash distribution from that for the generalized gust by purely algebraic manipulations.

The expansion of Eqs. (5) and (6) involve only the expansion of the Bessel function $J_k(\Lambda)$ for $k = 0, 1$. A Taylor expansion gives

$$\begin{aligned} J_k(\Lambda) = J_k(\lambda^i + \lambda) &= \sum_{n=0}^{\infty} (\lambda^n/n!) J_k^{(n)}(\lambda^i) \\ &= \sum_{n=0}^{\infty} i^n J_k^{(n)}(\lambda^i)(-i\lambda)^n/n! \end{aligned} \tag{8}$$

where $J_k^{(n)}$ is the nth derivative of J_k.

If we now simply insert Eq. (8) into Eqs. (5) and (6) and identify the coefficients of $(-i\lambda)^n/n!$, we find the lift and moment for the upwash

$$v(x,t) = v_0 e^{ivt}(x/c)^n \tag{9}$$

The results are

$$\begin{aligned} L(t) = 2\pi\beta^{-1}\rho_\infty cVv_0 e^{ivt}\Big\{&i^n[J_0^{(n)}(\lambda^i) - iJ_1^{(n)}(\lambda^i)] \times \\ &[C(\omega^i)J(\lambda^i) + iJ_1(\lambda^i)] + \\ &\frac{\omega i^{n+1}}{n+1}[J_0(\lambda^i)J_1^{(n+1)}(\lambda^i) - J_1(\lambda^i)J_0^{(n+1)}(\lambda^i)]\Big\} \end{aligned} \tag{10}$$

$$\begin{aligned} M(t) = \pi\beta^{-1}\rho_\infty c^2 Vv_0 e^{ivt}\Big\{&i^n[J_0^{(n)}(\lambda^i) - iJ_1^{(n)}(\lambda^i)] \times \\ &[[[C(\omega^i) - 1]][[J_0(\lambda^i) - J_2(\lambda^i)]] - 2iJ_1(\lambda^i)C(\omega^i)] - \\ &\frac{i i^{n+1}}{n+1}[J_0^{(n+1)}(\lambda^i)[[\omega J_0(\lambda^i) - \omega J_2(\lambda^i) - 2J_1(\lambda^i)]] + \\ &2J_1^{(n+1)}(\lambda^i)[[\omega J_1(\lambda^i) + J_0(\lambda^i)]]] - \\ &\frac{2\omega i^{n+2}}{(n+1)(n+2)}[J_0^{(n+2)}(\lambda^i)J_1(\lambda^i) - J_1^{(n+2)}(\lambda^i)J_0(\lambda^i)]\Big\} \end{aligned} \tag{11}$$

The cases $n = 0$, 1 were given in closed form in Ref. 5, in Eqs. (12, 13, 19, and 21). The preceding expressions reduce to those results, after a misprint in Eq. (21) of Ref. 5 is corrected. In the last term on the right of that equation, the factor $1 + (\omega/3)$ should be replaced by

$$(M^2\lambda^i)^{-1} + (\omega/3)$$

Results for larger values of n can be obtained by using recurrence relations to express the derivatives of the Bessel functions in terms of the Bessel functions themselves. Only the derivatives of J_0 and J_1 are required in Eqs. (10) and (11), and useful relations for their derivatives are

$$J_1^{(n)} = -J_0^{(n+1)}$$

$$\begin{aligned} i^n J_0^{(n)} &= \frac{2^{-n} n! J_0}{[(n/2)!]^2} + 2^{-n} n! (-1)^{n/2} \sum_{m=1+n/2}^{n} \frac{2(-1)^m J_{2m-n}}{(n-m)!m!}, \quad n \text{ even} \\ &= 2^{-n} n! i (-1)^{(n-1)/2} \sum_{m=(n+1)/2}^{n} \frac{2(-1)^m J_{2m-n}}{(n-m)!m!}, \quad n \text{ odd} \end{aligned}$$

Further reduction can be achieved by using recurrence relations to reduce all Bessel functions to a combination of J_0 and J_1.

Incompressible Flow

For incompressible flow, Osborne's theory reduces to the exact theory, so the incompressible results can be obtained by letting $\lambda^i \to 0$ in Eqs. (10) and (11). They can also be derived directly by using the incompressible generalized gust expressions from Refs. 2 and 3, which are

$$L(t) = 2\pi\rho_\infty cVv_0 e^{ivt}[J(\lambda)C(\omega) + i(\omega/\lambda)J_1(\lambda)] \tag{12}$$

$$M(t) = \pi\rho_\infty c^2 Vv_0 e^{ivt}\{J(\lambda)[C(\omega) - 1] + (\omega/\lambda)J_0(\lambda) + (2/\lambda)(1 - \omega/\lambda)J_1(\lambda)\} \tag{13}$$

It can be verified that these also follow from Eqs. (5) and (6) by making $\lambda^i \to 0$, $\omega^i \to \omega$.

To obtain the incompressible formulas from Eqs. (12) and (13), the power series for J_k is used as

$$J_k(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{k+2n}}{n!(n+k)!} = \sum_{n=0}^{\infty} (-1)^n \frac{(-iz)^{k+2n}(i/2)^{k+2n}}{n!(n+k)!} \tag{14}$$

This is inserted in Eqs. (12) and (13) with $z = \lambda$, and the coefficient of $(-i\lambda)^n/n!$ is extracted. To obtain the formulas from the $\lambda^i \to 0$ limit of Eqs. (10) and (11), the $z \to 0$ limit of the mth derivative of Eq. (14) is needed. The odd order Bessel functions have odd powers so only their odd-order derivatives are nonzero in the limit; similarly, the even order Bessel functions have only even order derivatives. The limits are

$$i^m J_0^{(m)}(\lambda^i) \to 2^{-m} m!/[(m/2)!]^2 \quad m \text{ even}$$

$$i^m J_1^{(m)}(\lambda^i) \to i2^{-m} m!/[(m-1)/2]![(m+1)/2]! \quad m \text{ odd}$$

These are inserted into Eqs. (10) and (11) together with $\lambda^i \to 0$, $\omega^i \to \omega$ in the other terms.

By either method the results are

$$\frac{L(t)}{2\pi\rho_\infty cVv_0 e^{ivt}} = \frac{2^{-n}n!}{[(n/2)!]^2}\left[C(\omega) + \frac{i\omega}{n+2}\right] \quad n \text{ even} \tag{15a}$$

$$= \frac{2^{-n}n!C(\omega)}{[(n-1)/2]![(n+1)/2]!} \quad n \text{ odd} \tag{15b}$$

$$\frac{M(t)}{\pi\rho_\infty c^2 Vv_0 e^{ivt}} = \frac{2^{-n}n!}{[(n/2)!]^2}\left[C(\omega) - \frac{n}{n+2}\right] \quad n \text{ even} \tag{16a}$$

$$= \frac{2^{-n}n!}{[(n-1)/2]![(n+1)/2]!}\left[C(\omega) - 1 - \frac{i\omega}{n+3}\right] n \text{ odd} \tag{16b}$$

These agree with the classical results for $n = 0$, 1, and give very simple expressions for $n > 1$.

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Technical Comments

Comment on "Stability of Rotating Stratified Fluids"

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IN a recent note Johnston[1] reported a new type of hydrodynamic instability observed in rotating flows with radially dependent density. He found that when a light gas (helium) is injected tangentially into a stationary gas of higher density (nitrogen) contained within a cylindrical test vessel that the resultant flow, as observed with aid of a laser-illuminated shadowgraph system, is unstable. The early stages of the instability are characterized by the appearance of nonaxisymmetric modes consisting of "two vortices which travel as well as rotate in the same direction as the main flow." The instability occurs under conditions where the product of the flow density and the square of its circulation is an increasing function of radius and hence, under conditions for which the classical Rayleigh–Synge criterion predicts stability for axisymmetric modes. The instability is observed to occur only when the density decreases with increasing radius in at least part of the flow region. No instabilities are found when a heavy gas is injected tangentially at the periphery of a gas core of lower density.

In attempting to explain these observations, Johnston[1] applied a modification of the Rayleigh–Synge criterion first proposed by Yih[2] and valid for flows where viscosity and thermal diffusivity play an important role in the instability growth mechanism. We wish to point out that such an application of Yih's criterion is incorrect since the criterion is applicable only to axisymmetric disturbances, while the instabilities observed are nonaxisymmetric as clearly shown in Fig. 2 of Johnston's paper. In addition, it is difficult to see how the dissipative effects of viscosity and thermal diffusivity can play a major role in the rapid disintegration process characterizing this instability phenomenon.

A correct interpretation of the observed instabilities may be obtained by use of a general stability criterion given by us in an earlier note.[3] This criterion represents a sufficiency condition for the stability of heterogeneous swirling flows against arbitrary infinitesimal disturbances of axial wave number k and azimuthal wave number m and is valid in the absence of viscous and gravitational effects. For the rotating flows examined by Johnston, the sufficiency condition for flow stability given in Ref. 3 reduces to the simplified form

$$(\rho r^3)^{-1}(d/dr)[\rho(r^2\Omega)^2] + (m/rk)^2[r\Omega^2(d/dr)(\ln\rho) - \tfrac{1}{4}r^2(d\Omega/dr)^2] \geq 0 \tag{1}$$

where Ω and ρ represent the radially dependent angular velocity and density, respectively. For axisymmetric disturbances ($m = 0$) the inequality is identical with the Rayleigh-Synge criterion while for nonaxisymmetric modes with $k = 0$, the flow will generally be stable only when the radial density gradient is positive and sufficiently large to overcome the destabilizing effect of the angular velocity gradient. The nonaxisymmetric instabilities observed in Johnston's experiments occur for negative radial density gradient and under conditions where the Rayleigh–Synge criterion is satisfied. One sees that these conditions correspond precisely to the case for which inequality (1) is violated for nonaxisymmetric modes only. Although small violations of criterion (1) need not necessarily imply instability, negative radial density gradients can generally be expected to lead to nonaxisymmetric modes of instability and will do so without the need to invoke the dissipative effects of viscosity. Finally, it is interesting to note from criterion (1) that heterogeneous rotating flows can be guaranteed stable for all infinitesimal disturbances when the radial density gradient is positive and the angular velocity gradient remains small. These are precisely the conditions encountered in centrifuges.

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